

F-DUGUNDJI SPACES, F-MILUTIN SPACES AND ABSOLUTE F-VALUED RETRACTS

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Dedicated to the memory of V.V. Fedorchuk

ABSTRACT. For every functional functor $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ in the category \mathbf{Comp} of compact Hausdorff spaces we define the notions of F -Dugundji and F -Milutin spaces, generalizing the classical notions of a Dugundji and Milutin spaces. We prove that the class of F -Dugundji spaces coincides with the class of absolute F -valued retracts. Next, we show that for a monomorphic continuous functor $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ admitting tensor products each Dugundji compact is an absolute F -valued retract if and only if the doubleton $\{0, 1\}$ is an absolute F -valued retract if and only if some points $a \in F(\{0\}) \subset F(\{0, 1\})$ and $b \in F(\{1\}) \subset F(\{0, 1\})$ can be linked by a continuous path in $F(\{0, 1\})$. We prove that for the functor Lip_k of k -Lipschitz functionals with $k < 2$, each absolute Lip_k -valued retract is openly generated. On the other hand the one-point compactification of any uncountable discrete space is not openly generated but is an absolute Lip_3 -valued retract. More generally, each hereditarily paracompact scattered compact space X of finite scattered height $n = \text{ht}(X)$ is an absolute Lip_k -valued retract for $k = 2^{n+2} - 1$.

1. INTRODUCTION

A classical Tietze-Urysohn Theorem [6, 2.1.8] says that each continuous function $f : X \rightarrow \mathbb{R}$ defined on a closed subset X of a normal topological space Y admits a continuous extension $\bar{f} : Y \rightarrow \mathbb{R}$. According to a classical theorem of Dugundji [5], for a closed subset X of a metrizable topological space Y and a locally convex linear topological space Z there is a linear operator $u : C(X, Z) \rightarrow C(Y, Z)$ extending each continuous function $f \in C(X, Z)$ to a function $\bar{f} \in C(Y, Z)$ with values in the closed convex hull $\overline{\text{conv}}(f(X))$ of $f(X)$ in Z . Operators with this property will be called *regular*.

Here by $C(X, Z)$ we denote the linear space of all continuous maps from X to Z . The linear space $C(X, \mathbb{R})$ of real-valued continuous functions on a topological space X is usually denoted by $C(X)$. If the space X is compact, then the linear space $C(X)$ carries a structure of a Banach lattice with respect to the sup-norm $\|f\| = \sup_{x \in X} |f(x)|$.

A natural temptation to unify Tietze-Urysohn and Dugundji Theorem fails as there are pairs (X, A) of compact Hausdorff (and hence normal topological) spaces $A \subset X$ admitting no regular linear extension operator $u : C(A) \rightarrow C(X)$. This circumstance led A. Pełczyński [14] to the idea of introducing the class of Dugundji compact spaces. Those are compact spaces X admitting for each embedding $X \hookrightarrow Y$ into a compact Hausdorff space Y a regular linear extension operator $u : C(X) \rightarrow C(Y)$.

The systematic study of the class of Dugundji compact spaces was started by A. Pełczyński in [14]. Soon, it was realized that Dugundji compact spaces can be characterized as absolute P -valued retracts for the functor $P : \mathbf{Comp} \rightarrow \mathbf{Comp}$ of probability measures in the category \mathbf{Comp} of compact Hausdorff spaces and their continuous maps. Let us recall that for a compact Hausdorff space X its space of probability measures PX is the subspace of the Tychonoff power $\mathbb{R}^{C(X)}$ consisting of all regular linear functionals $\mu : C(X) \rightarrow \mathbb{R}$ (the regularity of μ means that $\mu(f) \in \overline{\text{conv}}(f(X))$). Each point $x \in X$ can be identified with the Dirac measure $\delta_x : C(X) \rightarrow \mathbb{R}$, assigning to each function $f \in C(X)$ its value $f(x)$ at x . The assignment $x \mapsto \delta_x$ defines a canonical embedding $\delta : X \rightarrow PX$ of X into its space of probability measures.

A compact Hausdorff space X is called an *absolute P -valued retract* if for each embedding $X \subset Y$ into a compact Hausdorff space Y there is a continuous map $f : Y \rightarrow PX$ extending the canonical embedding $\delta : X \rightarrow PX$, see [8] for more details.

A breakthrough in understanding the structure of Dugundji compacta was made by R. Haydon [10] who proved that the class of Dugundji compacta coincides with the class $\text{AE}(0)$ of compact absolute extensors in dimension zero. We say that a topological space X is an *absolute extensor in dimension n* if each continuous map $f : B \rightarrow X$ defined on a closed subspace B of a compact Hausdorff space A of dimension $\dim(A) \leq n$ admits a continuous extension $\bar{f} : A \rightarrow X$. By $\text{AE}(n)$ we shall denote the class of compact absolute extensors in dimension n .

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Theorem 1.1 (Haydon). *For a compact Hausdorff space X the following conditions are equivalent:*

- (1) X is a Dugundji compact space;
- (2) X is an absolute P -valued retract;
- (3) X is an absolute extensor in dimension 0.

The implication (3) \Rightarrow (1) of this theorem is usually proved with help of Milutin compact spaces, see [9]. Let us recall [9] that a compact Hausdorff space X is *Milutin* if there is a continuous surjective map $f : K \rightarrow X$ from a Cantor cube $K = \{0, 1\}^\kappa$, admitting a regular averaging operator $u : C(K) \rightarrow C(X)$, i.e., a regular linear operator such that $u(\varphi \circ f) = \varphi$ for any $\varphi \in C(X)$. In [13] Milutin proved that the unit interval $\mathbb{I} = [0, 1]$ is Milutin and derived from this fact that each Dugundji compact space is Milutin. The converse is not true as shown by the example of the hyperspace $\exp_2(\{0, 1\}^{\mathbb{N}_2})$ which is Milutin but not Dugundji, see [9, 6.7].

Theorem 1.1 shows that Dugundji compact spaces are tightly connected with the functor of probability measures P (this was observed and widely exploited by Ščepin in [22]). The relations of the class of Dugundji spaces to some other functors was studied by Alkinson and Valov [1], [25].

In this paper for any functional functor $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ we define the notions of F -Dugundji and F -Milutin compact spaces and will characterize these spaces in terms of extension and averaging operators between the spaces of continuous functions thus generalizing Theorem 1.1 to other functors. In particular, we shall prove that the class of F -Dugundji compact spaces coincides with the class of absolute F -valued retracts. In Sections 3 and 4 for certain (concrete functional) functors we shall study the class of absolute F -valued retracts and its relation to the classes of Dugundji compact spaces and of openly generated compacta.

2. F -DUGUNDJI AND F -MILUTIN COMPACT SPACES

All topological spaces considered in this paper are assumed to be Hausdorff. Undefined notions from the theory of functors in the category \mathbf{Comp} can be found in the monograph [23].

For a compact space X by $C(X)$ we denote the Banach lattice of all continuous functions endowed with the norm $\|\varphi\| = \sup_{x \in X} |\varphi(x)|$. Any (not necessarily continuous) function $\mu : C(X) \rightarrow \mathbb{R}$ will be called a *functional* on $C(X)$. The space $\mathbb{R}^{C(X)}$ of all functionals will be endowed with the Tychonoff product topology. For every $x \in X$ the Dirac measure $\delta_X(x) \in \mathbb{R}^{C(X)}$ is the functional assigning to each function $\varphi \in C(X)$ its value $\varphi(x)$ at x .

Any continuous map $f : X \rightarrow Y$ between compact spaces induces a linear operator $f^* : C(Y) \rightarrow C(X)$, $f^* : \varphi \mapsto \varphi \circ f$, between the corresponding function spaces, called the *dual of f* . The *second dual operator* of f is the function $f^{**} : \mathbb{R}^{C(X)} \rightarrow \mathbb{R}^{C(Y)}$ assigning to each functional $\mu \in \mathbb{R}^{C(X)}$ the functional $f^{**}(\mu) \in \mathbb{R}^{C(Y)}$, $f^{**}(\mu) : \varphi \mapsto \mu(\varphi \circ f)$. Letting $\mathbb{R}^{C(f)} := f^{**}$, we can consider the construction $\mathbb{R}^{C(\cdot)} : \mathbf{Comp} \rightarrow \mathbf{Tych}$ as a covariant functor from the category \mathbf{Comp} to the category \mathbf{Tych} of Tychonoff spaces and their continuous maps. The functor $\mathbb{R}^{C(\cdot)}$ contains the Dirac functor as a subfunctor. The *Dirac functor* $\delta : \mathbf{Comp} \rightarrow \mathbf{Comp}$ assigns to each compact space X the closed subspace $\delta(X) = \{\delta_X(x) : x \in X\} \subset \mathbb{R}^{C(X)}$ consisting of the Dirac measures on X . It is clear that the Dirac functor δ is isomorphic to the identity functor.

The Tietze-Urysohn Theorem implies that for any injective continuous map $f : X \rightarrow Y$ between compacta the dual map $f^* : C(Y) \rightarrow C(X)$ is surjective and then the second dual map $f^{**} : \mathbb{R}^{C(X)} \rightarrow \mathbb{R}^{C(Y)}$ is a topological embedding. Consequently, for each closed subset $A \subset X$ and the identity embedding $i_A : A \rightarrow X$ we can identify the functional space $\mathbb{R}^{C(A)}$ with the subspace $i_A^{**}(\mathbb{R}^{C(A)}) \subset \mathbb{R}^{C(X)}$. The same convention will concern also other functors F : writing $a \in FA \subset FX$ we shall have in mind that $a \in Fi_A(FA) \subset FX$.

By a *functional functor* we shall understand any subfunctor $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ of the functor $\mathbb{R}^{C(\cdot)} : \mathbf{Comp} \rightarrow \mathbf{Tych}$, containing the Dirac functor $\delta \subset F$. Well-known examples of functional functors are the functor of probability measures P [8] and the functor of idempotent measures I [26]. Many known functors in the category \mathbf{Comp} are isomorphic to functional functors, see [16], [17].

The notion of a regular operator $u : C(X) \rightarrow C(Y)$ appearing in the definitions of Dugundji and Milutin spaces is a partial case of the notion of an F -regular operator for a functional functor F . By an *operator* between function spaces $C(X)$ and $C(Y)$ we shall understand any (not necessarily linear or continuous) function $u : C(X) \rightarrow C(Y)$. Each operator $u : C(X) \rightarrow C(Y)$ induces the dual operator $u^* : \mathbb{R}^{C(X)} \rightarrow \mathbb{R}^{C(Y)}$ assigning to each functional $\mu : C(Y) \rightarrow \mathbb{R}$ the functional $u^*(\mu) \in \mathbb{R}^{C(X)}$, $u^*(\mu) : \varphi \mapsto \mu \circ u(\varphi)$. It is easy to check that the dual operator u^* is continuous with respect to the Tychonoff product topologies on the functional spaces $\mathbb{R}^{C(X)}$ and $\mathbb{R}^{C(Y)}$.

Definition 2.1. For a functional functor $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ and compact spaces X, Y , an operator $u : C(X) \rightarrow C(Y)$ is called *F -regular* if for any $y \in Y$ the functional $u^*(\delta_Y(y))$ belongs to $FX \subset \mathbb{R}^{C(X)}$.

Observe that an operator $u : C(X) \rightarrow C(Y)$ is regular and linear if and only if it is P -regular for the functor of probability measures P .

Next we generalize the notions of Dugundji and Milutin compact spaces introducing a functorial parameter in their definitions.

Definition 2.2. Let $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ be a functional functor. A compact space X is defined to be

- *F-Milutin* if there exists a surjective map $f : K \rightarrow X$ from a Cantor cube $K = \{0, 1\}^\kappa$ admitting an F -regular averaging operator $u : C(K) \rightarrow C(X)$;
- *F-Dugundji* if there exists an injective map $f : X \rightarrow K$ to a Tychonoff cube $K = [0, 1]^\kappa$ admitting an F -regular extension operator $u : C(X) \rightarrow C(K)$.

The notions of an extension and averaging operators is unified by the notion of an exave operator introduced by Pełczyński [14].

An operator $u : C(X) \rightarrow C(Y)$ is called an *f-exave* for f if $f^* \circ u \circ f^* = f^*$. If f is injective (resp. surjective), then the equality $f^* \circ u \circ f^* = f^*$ is equivalent to $f^* \circ u = \text{id}_{C(X)}$ (resp. $u \circ f^* = \text{id}_{C(Y)}$), in which case u is called an *extension* (resp. *averaging*) operator for f .

Theorem 2.3. For a functional functor $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ and a map $f : X \rightarrow Y$ between compact Hausdorff spaces X, Y the following conditions are equivalent:

- (1) there exists an F -regular exave operator $u : C(X) \rightarrow C(Y)$ for the map f ;
- (2) there exists a continuous map $s : Y \rightarrow FX$ such that $Ff \circ s(y) = \delta_Y(y)$ for every $y \in f(X) \subset Y$.

Proof. To prove that (1) \Rightarrow (2), assume that $u : C(X) \rightarrow C(Y)$ is an F -regular exave operator $u : C(X) \rightarrow C(Y)$ for the map f . Then its dual operator $u^* : \mathbb{R}^{C(Y)} \rightarrow \mathbb{R}^{C(X)}$ is a continuous map such that $u^* \circ \delta_Y(Y) \subset FX$. Consider the map $s = u^* \circ \delta_Y : Y \rightarrow FX$ and take any point $y \in f(X)$. Choose any point $x \in f^{-1}(y)$. Since u is an exave, $f^* \circ u \circ f^* = f^*$, which implies $f^{**} \circ u^* \circ f^{**} = f^{**}$ where $f^{**} : \mathbb{R}^{C(X)} \rightarrow \mathbb{R}^{C(X)}$ is the dual operator to $f^* : C(Y) \rightarrow C(X)$. Taking into account that F is a subfunctor of the functor $\mathbb{R}^{C(\cdot)}$, we conclude that $Ff = f^{**}|_{FX}$. It is easy to check that $f^{**}(\delta_X(x)) = \delta_Y(f(x)) = \delta_Y(y)$, which implies that

$$\delta_Y(y) = f^{**}(\delta_X(x)) = f^{**} \circ u^* \circ f^{**}(\delta_X(x)) = f^{**} \circ u^*(\delta_Y(y)) = Ff \circ u^*(\delta_Y(y)) = Ff \circ s(y).$$

To prove that (2) \Rightarrow (1), assume that $s : Y \rightarrow FX$ is a continuous map such that $Ff \circ s(y) = \delta_Y(y)$ for every $y \in f(X) \subset Y$. Define an operator $u : C(X) \rightarrow C(Y)$ assigning to each continuous function $\varphi \in C(X)$ the continuous function $u(\varphi) : Y \rightarrow \mathbb{R}$, $u(\varphi) : y \mapsto s(y)(\varphi)$. The continuity of $u(\varphi)$ follows from the continuity of the function $s : Y \rightarrow FX \subset \mathbb{R}^{C(X)}$ and the continuity of the evaluation operator $\delta_\varphi : \mathbb{R}^{C(X)} \rightarrow \mathbb{R}$, $\delta_\varphi : \mu \mapsto \mu(\varphi)$.

Let us check that the operator $u : C(X) \rightarrow C(Y)$ is F -regular. Consider the dual operator $u^* : \mathbb{R}^{C(Y)} \rightarrow \mathbb{R}^{C(X)}$ and fix any point $y \in Y$. Observe that the functional $u^*(\delta_Y(y)) \in \mathbb{R}^{C(X)}$ assigns to each function $\varphi \in C(X)$ the real number $\delta_Y(y)(u(\varphi)) = u(\varphi)(y) = s(y)(\varphi)$, which implies that $u^* \circ \delta_Y(y) = s(y) \in FX$. This means that the operator u is F -regular.

Finally, we check that u is an exave for the map f , i.e., $f^* \circ u \circ f^* = f^*$ where $f^* : C(Y) \rightarrow C(X)$ is the dual operator induced by f . Given any function $\varphi \in C(Y)$ and any $x \in X$, we need to check that $f^* \circ u \circ f^*(\varphi)(x) = f^*(\varphi)(x)$. For this let $y = f(x)$ and observe that

$$f^* \circ u \circ f^*(\varphi)(x) = u(\varphi \circ f)(f(x)) = s(f(x))(\varphi \circ f) = Ff(s(f(x)))(\varphi) = \delta_Y(f(x))(\varphi) = \varphi(f(x)) = f^*(\varphi)(x).$$

Hence u is an exave for f and the theorem is proved. \square

Theorem 2.3 implies the following characterizations of F -Milutin and F -Dugundji compact spaces.

Theorem 2.4. For a functional functor $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ and a compact Hausdorff space X the following conditions are equivalent:

- (1) X is F -Milutin;
- (2) there exist a surjective map $f : K \rightarrow X$ from a Cantor cube $K = [0, 1]^\kappa$ and a continuous map $s : X \rightarrow FK$ such that $Ff \circ s(x) = \delta_X(x)$ for every $x \in X$.

Theorem 2.5. For a functional functor $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ and a compact space X the following conditions are equivalent:

- (1) X is F -Dugundji;
- (2) every embedding $X \subset Y$ into a compact Hausdorff space Y admits an F -regular extension operator $u : C(X) \rightarrow C(Y)$;
- (3) for every embedding $X \subset Y$ into a compact Hausdorff space Y there exists a continuous map $s : Y \rightarrow FX$ such that $s(x) = \delta_Y(x)$ for all $x \in X \subset Y$;
- (4) for some embedding $X \subset K$ into a Tychonoff cube $K = [0, 1]^\kappa$ there exists a continuous map $s : K \rightarrow FX$ such that $s(x) = \delta_K(x)$ for all $x \in X \subset K$.

Proof. The equivalences (2) \Leftrightarrow (3) and (1) \Leftrightarrow (4) follow from Theorem 2.3 while (3) \Rightarrow (4) is trivial. To prove that (4) \Rightarrow (3), assume that for some embedding $X \subset K$ into a Tychonoff cube $K = [0, 1]^\kappa$ there exists a continuous map $s : K \rightarrow FX$ such that $s(x) = \delta_K(x)$ for all $x \in X \subset K$. Let $X \subset Y$ be any embedding into a compact space Y . By Tietze-Urysohn Theorem, the identity map $X \rightarrow X \subset K$ admits a continuous extension $f : Y \rightarrow K$. Then the map $s \circ f : Y \rightarrow FX$ has the required property: $s \circ f(x) = s(x) = \delta_Y(x)$ for every $x \in X$. \square

3. ABSOLUTE F -VALUED RETRACTS AND ABSOLUTE F -MILUTIN SPACES

Observe that the last conditions in Theorems 2.5, 2.4 have sense for any (not necessarily functional) functor F . So, we have chosen these conditions as a base for the definitions of absolute F -valued retracts and absolute F -Milutin spaces. In the following definition, given a functor $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ and a closed subset $A \subset X$ of a compact Hausdorff space X we write $a \in FA \subset FX$ for some element $a \in FX$ if $a \in Fi_A^X(FA) \subset FX$ where $i_A^X : A \rightarrow X$ is the identity embedding.

Definition 3.1. Given a functor $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ we define a compact Hausdorff space X to be

- an *absolute F -valued retract* if for any embedding $X \subset Y$ into a compact space Y there is a map $r : Y \rightarrow FX$ such that $r(x) \in F(\{x\}) \subset FX$ for every point $x \in X$;
- an *absolute F -Milutin space* if there are a surjective map $f : K \rightarrow X$ from a Cantor cube $K = \{0, 1\}^\kappa$ and a map $s : X \rightarrow FK$ such that $s(x) \in F(f^{-1}(x)) \subset FK$ for all $x \in X$.

We shall say that a functor $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ *preserves preimages (over points)* if for any surjective map $f : X \rightarrow Y$ between compact spaces and any closed (one-point) set $A \subset Y$ we get $(Ff)^{-1}(FA) = F(f^{-1}(A))$ (which actually means that $(Ff)^{-1}(Fi_A^Y(FA)) = Fi_{f^{-1}(A)}^X(F(f^{-1}(A))) \subset FX$).

Theorems 2.5, 2.4 imply the following corollary.

Corollary 3.2. *Let $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ be a functional functor preserving singletons. A compact space X is*

- (1) *F -Dugundji if and only if X is an absolute F -valued retract;*
- (2) *F -Milutin if X is absolute F -Milutin.*

Moreover, if the functional functor F preserves preimages over points, then a compact space X is F -Milutin if and only if X is absolute F -Milutin.

Remark 3.3. In [1], [25] partial cases of Corollary 3.2 were proved for some concrete functional functors F .

For a functor $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ by $\mathbf{AR}[F]$ we shall denote the class of all compact absolute F -valued retracts. In the remaining part of the paper we shall address the following problem motivated by Theorem 1.1 and Corollary 3.2.

Problem 3.4. *Given a functor $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$, detect compact spaces that belong to the class $\mathbf{AR}[F]$.*

This problem is not new and has been considered in [22], [4], [2], [25]. An information about the classes $\mathbf{AR}[F]$ can be helpful because of the following simple fact.

Proposition 3.5. *Let $F, F' : \mathbf{Comp} \rightarrow \mathbf{Comp}$ be two functors. If F admits a natural transformation into F' , then $\mathbf{AR}[F] \subset \mathbf{AR}[F']$.*

For normal functors F the upper bound on the classes $\mathbf{AR}[F]$ was found by Ščepin [22].

Theorem 3.6 (Ščepin). $\mathbf{AR}[F] \subset \mathbf{AE}(0) = \mathbf{AR}[P]$ for any normal functor $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$.

Next, we find a condition on a functor F guaranteeing that $\mathbf{AE}(0) \subset \mathbf{AR}[F]$.

Proposition 3.7. *Let F be a functor. If each Tychonoff cube is a absolute F -Milutin space, then each Dugundji compact space is an absolute F -Milutin absolute F -valued retract. Consequently, $\mathbf{AE}(0) \subset \mathbf{AR}[F]$.*

Proof. Let X be a Dugundji compact space. To show that X is an absolute F -Milutin absolute F -valued retract, fix any embedding $X \subset Y$ into a compact space Y . Let $Y \subset K$ be an embedding of Y into a Tychonoff cube K . Since the Tychonoff cube K is absolute F -Milutin, there exist a continuous map $g : C \rightarrow K$ from a Cantor cube $C = \{0, 1\}^\kappa$ and a continuous map $s : K \rightarrow FK$ such that $s(x) \in F(g^{-1}(x)) \subset FK$ for each $x \in K$.

Consider the closed subset $Z = g^{-1}(X)$ in the Cantor cube C . By Haydon's Theorem 1.1, the Dugundji compact space X is an absolute extensor in dimension zero. Consequently, the map $g|_Z : Z \rightarrow X$ admits a continuous extension $\bar{g} : C \rightarrow X$. Consider the map $s|_X : X \rightarrow FK$ and observe that for every $x \in X$ we get

$$s(x) \in F(g^{-1}(x)) \subset F(\bar{g}^{-1}(x)) \subset FK.$$

Consequently, the maps $\bar{g} : C \rightarrow X$ and $s|_X : X \rightarrow FK$ witness that X is absolute F -Milutin.

On the other hand, the map $r = F\bar{g} \circ s|Y : Y \rightarrow FX$ witnesses that X is an absolute F -valued retract because

$$r(x) = F\bar{g}(s(x)) \in F\bar{g}(F(\bar{g}^{-1}(x))) \subset F(\{x\}) \subset FX$$

for every $x \in X$. \square

We shall say that a functor $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ admits a *tensor product* if for each cardinal κ and each family of compacta $(X_\alpha)_{\alpha \in \kappa}$ there exists a continuous map $\otimes_{(X_\alpha)_{\alpha \in \kappa}} : \prod_{\alpha \in \kappa} FX_\alpha \rightarrow F(\prod_{\alpha \in \kappa} X_\alpha)$ which is natural by each argument. The latter means that for any maps $f_\alpha : X_\alpha \rightarrow Y_\alpha$, $\alpha \in \kappa$, between compact spaces and any $\beta \in \kappa$ the following diagram commutes:

$$\begin{array}{ccc} & FX_\beta & \\ \text{pr}_\beta \nearrow & & \nwarrow F\text{pr}_\beta \\ \prod_{\alpha \in \kappa} FX_\alpha & \xrightarrow{\otimes_{(X_\alpha)_{\alpha \in \kappa}}} & F(\prod_{\alpha \in \kappa} X_\alpha) \\ \downarrow \prod_{\alpha \in \kappa} Ff_\alpha & & \downarrow F(\prod_{\alpha \in \kappa} f_\alpha) \\ \prod_{\alpha \in \kappa} FY_\alpha & \xrightarrow{\otimes_{(Y_\alpha)_{\alpha \in \kappa}}} & F(\prod_{\alpha \in \kappa} Y_\alpha) \end{array}$$

According to [20], for any functors $F_1, F_2 : \mathbf{Comp} \rightarrow \mathbf{Comp}$ admitting tensor products the composition $F_1 \circ F_2$ admits a tensor product too. It is known that each monadic functor (i.e., a functor that can be completed to a monad) admits a tensor product (see [23, §3.4]). In particular, the functors of probability measures P [8] and idempotent measures [26] are monadic and hence admit a tensor product. On the other hand, the functor $\exp \circ \exp$ of double hyperspace admits a tensor product but fails to be monadic, see [20].

Theorem 3.8. *Let $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ be a functor admitting a tensor product. If the unit interval $\mathbb{I} = [0, 1]$ is an absolute F -Milutin space, then every Tychonoff cube is an absolute F -Milutin space and $\text{AE}(0) \subset \text{AR}[F]$.*

Proof. By Proposition 3.7, it suffices to prove that each Tychonoff cube \mathbb{I}^κ is an absolute F -Milutin space. Since \mathbb{I} is an absolute F -Milutin space, there exist a surjective map $g : C \rightarrow \mathbb{I}$ from a Cantor cube C and a map $s : \mathbb{I} \rightarrow FC$ such that $s(x) \in F(g^{-1}(x)) \subset FC$ for all $x \in \mathbb{I}$. Consider the κ -th power $\bar{g} : C^\kappa \rightarrow \mathbb{I}^\kappa$, $\bar{g} : (x_\alpha)_{\alpha \in \kappa} \mapsto (g(x_\alpha))_{\alpha \in \kappa}$, of the map g and the κ -th power $\bar{s} : \mathbb{I}^\kappa \rightarrow (FC)^\kappa$ of the map s .

Fix a tensor product \otimes for the functor F and let $\otimes_{(C)_{\alpha \in \kappa}} : (FC)^\kappa \rightarrow F(C^\kappa)$ be its component. We claim that the map $r = \otimes_{(C)_{\alpha \in \kappa}} \circ \bar{s} : \mathbb{I}^\kappa \rightarrow F(C^\kappa)$ witnesses that the Tychonoff cube \mathbb{I}^κ is absolute F -Milutin. Indeed, given any point $x = (x_\alpha)_{\alpha \in \kappa} \in \mathbb{I}^\kappa$, consider the element $\bar{s}(x) \in \prod_{\alpha \in \kappa} F(g^{-1}(x_\alpha)) \subset \prod_{\alpha \in \kappa} FC$. By the naturality of the tensor product,

$$r(x) = \otimes_{(g^{-1}(x_\alpha))_{\alpha \in \kappa}} \circ \bar{s}(x) \in F\left(\prod_{\alpha \in \kappa} g^{-1}(x_\alpha)\right) = F(\bar{g}^{-1}(x)) \subset F(C^\kappa).$$

\square

In light of Theorem 3.8, it is natural to pose a problem of recognizing functors F for which the unit interval $\mathbb{I} = [0, 1]$ is absolute F -Milutin. We shall give an answer to this problem for functionally continuous monomorphic functors which admit a tensor product.

A functor $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ is called *functionally continuous* if for each compact spaces X, Y the map $F_{X,Y} : C(X, Y) \rightarrow C(FX, FY)$, $F_{X,Y} : f \mapsto Ff$, is continuous. Here by $C(X, Y)$ we denote the space of all continuous functions from X to Y , endowed with the compact-open topology. By [23, 2.2.3], a monomorphic [epimorphic] functor $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ is functionally continuous if [and only if] it is continuous (i.e., preserves limits of inverse spectra). We recall that a functor $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ is *monomorphic* (resp. *epimorphic*) if F preserves injective (resp. surjective) maps.

Theorem 3.9. *For a monomorphic functionally continuous functor $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$, admitting a tensor product, the following conditions are equivalent:*

- (1) *the closed interval $[0, 1]$ is an absolute F -Milutin space;*
- (2) *every Tychonoff cube is an absolute F -Milutin space;*
- (3) *$\text{AE}(0) \subset \text{AR}[F]$;*
- (4) *the doubleton $\{0, 1\}$ is an absolute F -valued retract;*
- (5) *there are points $a \in F(\{0\}) \subset F(\{0, 1\})$ and $b \in F(\{1\}) \subset F(\{0, 1\})$ which can be linked by a continuous path in $F(\{0, 1\})$.*

- (6) *there is a point $a \in F(\{0\})$ such that for the embeddings $i_0 : \{0\} \rightarrow \{0\} \subset \{0, 1\}$ and $i_1 : \{0\} \rightarrow \{1\} \subset \{0, 1\}$ the points $Fi_0(a)$ and $Fi_1(a)$ can be linked by a continuous path in $F(\{0, 1\})$;*

Proof. We shall prove the implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6) \Rightarrow (1)$.

The implications $(1) \Rightarrow (2) \Rightarrow (3)$ have been proved in Theorem 3.8 and Proposition 3.7 while $(3) \Rightarrow (4)$ is trivial.

$(4) \Rightarrow (5)$ Assume that the doubleton $\{0, 1\}$ is an absolute F -valued retract. Then there is a continuous map $\gamma : [0, 1] \rightarrow F(\{0, 1\})$ such that $\gamma(x) \in F(\{x\}) \subset F(\{0, 1\})$ for every $x \in \{0, 1\}$. The map γ can be considered as a continuous path in $F(\{0, 1\})$ linking the points $a = \gamma(0) \in F(\{0\}) \subset F(\{0, 1\})$ and $b = \gamma(1) \in F(\{1\}) \subset F(\{0, 1\})$.

$(5) \Rightarrow (6)$ Assume that some points $a_0 \in F(\{0\}) \subset F(\{0, 1\})$ and $a_1 \in F(\{1\}) \subset F(\{0, 1\})$ can be linked by a continuous path $\gamma : [0, 1] \rightarrow F(\{0, 1\})$ such that $\gamma(i) = a_i$ for each $i \in \{0, 1\}$. Since $a_0 \in F(\{0\}) \subset F(\{0, 1\})$, there is a point $a \in F(\{0\})$ such that $a_0 = Fi_0(a)$ where $i_0 : \{0\} \rightarrow \{0, 1\}$ is the identity embedding. Let $i_1 : \{0\} \rightarrow \{1\} \subset \{0, 1\}$ be the other embedding of $\{0\}$ into $\{0, 1\}$.

Consider the retraction $r : \{0, 1\} \rightarrow \{1\} \subset \{0, 1\}$ and observe that the map $Fr : F(\{0, 1\}) \rightarrow F(\{1\}) \subset F(\{0, 1\})$ is a retraction of the space $F(\{0, 1\})$ onto its subspace $F(\{1\}) = i_{\{1\}}^{\{0, 1\}}(F(\{1\}))$. It follows that $\tilde{\gamma} = Fr \circ \gamma : \mathbb{I} \rightarrow F(\{1\}) \subset F(\{0, 1\})$ is a continuous path linking the points $\tilde{\gamma}(0) = Fr(a_0) = FrFi_0(a) = F(r \circ i_0)(a) = Fi_1(a)$ and $\tilde{\gamma}(1) = Fr(a_1) = a_1$. Joining the paths γ and $\tilde{\gamma}$ together, we can construct a continuous path in $F(\{0, 1\})$ linking the points $Fi_0(a) = a_0$ and $Fi_1(a)$.

$(6) \Rightarrow (1)$ Assume that for some $a \in F(\{0\})$ and the embeddings $i_0 : \{0\} \rightarrow \{0\} \subset \{0, 1\}$ and $i_1 : \{0\} \rightarrow \{1\} \subset \{0, 1\}$ the points $a_0 = Fi_0(a)$ and $a_1 = Fi_1(a)$ can be linked by a continuous path $\gamma : [0, 1] \rightarrow F(\{0, 1\})$ such that $\gamma(i) = a_i$ for $i \in \{0, 1\}$. For every compact space X the functional continuity of the functor F implies the continuity of the map $F_X : C(\{0, 1\}, X) \rightarrow C(F\{0, 1\}, FX)$, $F_X : f \mapsto Ff$, which implies the continuity of the map

$$\Phi_X : F(\{0, 1\}) \times C(\{0, 1\}, X) \rightarrow FX, \quad \Phi_X : (c, f) \mapsto Ff(c).$$

Identify the square $X \times X$ of X with the function space $C(\{0, 1\}, X)$, assigning to each pair $(x_0, x_1) \in X \times X$ the function $f_{x_0, x_1} : \{0, 1\} \rightarrow X$, $f_{x_0, x_1} : i \mapsto x_i$. The continuity of the function Φ_X implies the continuity of the function

$$\Psi_X : X \times X \times \mathbb{I} \rightarrow FX, \quad \Psi_X : (x, y, t) \mapsto \Phi_X(\gamma(t), f_{x, y}) = Ff_{x, y}(\gamma(t)).$$

For every $x \in X$ consider the embeddings $f_x : \{0\} \rightarrow \{x\} \subset X$ and $Ff_x : F(\{0\}) \rightarrow FX$. For every $x, y \in X$ the equalities $f_{x, y} \circ i_0 = f_x$ and $f_{x, y} \circ i_1 = f_y$ imply that

$$\Psi_X(x, y, 0) = Ff_{x, y}(\gamma(0)) = Ff_{x, y}Fi_0(a) = F(f_{x, y} \circ i_0)(a) = Ff_x(a)$$

and

$$\Psi_X(x, y, 1) = Ff_{x, y}(\gamma(1)) = Ff_{x, y}Fi_1(a) = F(f_{x, y} \circ i_1)(a) = Ff_y(a).$$

For every $n \in \omega$ and $k \in \{1, \dots, 2^n\}$ consider the interval $\mathbb{J}_{n, k} = [\frac{k-1}{2^n}, \frac{k}{2^n}] \subset \mathbb{I}$ and its closed neighborhood $\mathbb{J}_{n, k} = [0, 1] \cap [\frac{k-1}{2^n} - \frac{1}{2^{n+2}}, \frac{k}{2^n} + \frac{1}{2^{n+2}}]$. We shall consider the finite family $\{\mathbb{J}_{n, k}\}_{k=1}^{2^n}$ as a compact space endowed with the discrete topology.

Let

$$\mathbb{J}_n = \bigcup_{k=1}^{2^n} \mathbb{J}_{n, k} \times \{\mathbb{J}_{n, k}\} \subset \mathbb{I} \times \{\mathbb{J}_{n, k}\}_{k=1}^{2^n}$$

be the topological sum of the family $\{\mathbb{J}_{n, k}\}_{k=1}^{2^n}$ and $\text{pr}_n : \mathbb{J}_n \rightarrow \mathbb{I}$, $\text{pr}_n : (t, i) \mapsto t$, be the coordinate projection. For every $t \in \mathbb{I}$ let $k_0 \leq k_1$ be unique numbers such that $\{k_0, k_1\} = \{k \in \{1, \dots, 2^n\} : t \in \mathbb{J}_{n, k}\}$ and let $t_0 = (t, \mathbb{J}_{n, k_0})$, $t_1 = (t, \mathbb{J}_{n, k_1})$ be the points composing the preimage $\text{pr}_n^{-1}(t) = \{t_0, t_1\}$. Observe that for every $t \in \mathbb{I} \setminus \bigcup_{k=1}^{2^n-1} (\mathbb{J}_{n, k} \cap \mathbb{J}_{n, k+1})$ the points t_0 and t_1 coincide.

Define a continuous map $s_n : \mathbb{I} \rightarrow F(\mathbb{J}_n)$ letting

$$s_n(t) = \begin{cases} \Psi_{\mathbb{J}_n}(t_0, t_1, \frac{t - \min \mathbb{J}_{n, k+1}}{\max \mathbb{J}_{n, k} - \min \mathbb{J}_{n, k+1}}), & \text{if } t \in \mathbb{J}_{n, k} \cap \mathbb{J}_{n, k+1} \text{ for some } 0 < k < 2^n, \\ \Psi_{\mathbb{J}_n}(t_0, t_1, 0) = \Psi_{X_n}(t_0, t_1, 1), & \text{otherwise,} \end{cases}$$

and observe that $s_n(t) \in F(\{t_0, t_1\}) \subset F\mathbb{J}_n$ for every $t \in \mathbb{I}$.

In the Tychonoff product $\mathbb{J}_\omega = \prod_{n \in \omega} \mathbb{J}_n$ consider the closed subset

$$K = \{(x_n) \in \mathbb{J}_\omega : \forall n, m \in \omega \quad \text{pr}_n(x_n) = \text{pr}_m(x_m)\}$$

and let $\text{pr} : K \rightarrow \mathbb{I}$ be the projection on \mathbb{I} defined by $\text{pr}((x_n)_{n \in \omega}) = \text{pr}_1(x_1)$ for $(x_n)_{n \in \omega} \in K$. It can be shown that K is a compact zero-dimensional space without isolated points, so K is homeomorphic to the Cantor cube $\{0, 1\}^\omega$

according to the Brouwer theorem [12, 7.4] characterizing the Cantor set. We claim that the map $\text{pr} : K \rightarrow \mathbb{I}$ witnesses that the interval $\mathbb{I} = [0, 1]$ is absolute F -Milutin.

By our assumption, the functor F admits a tensor product \otimes . Let

$$\otimes_{(\mathbb{J}_n)_{n \in \omega}} : \prod_{n \in \omega} F\mathbb{J}_n \rightarrow F\left(\prod_{n \in \omega} \mathbb{J}_n\right) = F\mathbb{J}_\omega$$

be its component for the family $(\mathbb{J}_n)_{n \in \omega}$.

Consider the continuous map $s : \mathbb{I} \rightarrow F\mathbb{J}_\omega$ defined by $s(t) = \otimes_{(\mathbb{J}_n)_{n \in \omega}} ((s_n(t))_{n \in \omega})$. Since the functor F is monomorphic, for the identity embedding $i_K : K \rightarrow \mathbb{J}_\omega$ the map $F i_K : FK \rightarrow F\mathbb{J}_\omega$ is a topological embedding, so we can (and will) identify FK with its image $F i_K(FK)$ in $F\mathbb{J}_\omega$. We claim that for every $t \in \mathbb{I}$ the point $s(t)$ is contained in the set $F(\text{pr}^{-1}(t)) = F(\prod_{n \in \omega} \text{pr}_n^{-1}(t)) \subset FK \subset F\mathbb{J}_\omega$.

By the definition of the map s_n , for every $n \in \omega$ the point $s_n(t)$ is contained in the set $F(\text{pr}_n^{-1}(t)) \subset F(\mathbb{J}_n)$ and then $(s_n(t))_{n \in \omega} \in \prod_{n \in \omega} F(\text{pr}_n^{-1}(t)) \subset \prod_{n \in \omega} F(\mathbb{J}_n)$. Let

$$\otimes_{(\text{pr}_n^{-1}(t))_{n \in \omega}} : \prod_{n \in \omega} F(\text{pr}_n^{-1}(t)) \rightarrow F\left(\prod_{n \in \omega} \text{pr}_n^{-1}(t)\right) = F(\text{pr}^{-1}(t)) \subset FK$$

be the component of the tensor product for the family $(\text{pr}_n^{-1}(t))_{n \in \omega}$. The naturality of the tensor product guarantees that

$$s(t) = \otimes_{(\mathbb{J}_n)_{n \in \omega}} ((s_n(t))_{n \in \omega}) = \otimes_{(\text{pr}_n^{-1}(t))_{n \in \omega}} (s_n(t))_{n \in \omega} \in F\left(\prod_{n \in \omega} \text{pr}_n^{-1}(t)\right) = F(\text{pr}^{-1}(t)) \subset FK.$$

So, $s : \mathbb{I} \rightarrow FK$ is a well-defined continuous map witnessing that the closed interval $\mathbb{I} = [0, 1]$ is an absolute F -Milutin space. \square

Combining Theorem 3.9 with Shchepin's Theorem 3.6, we get:

Corollary 3.10. *If a normal functor $F : \text{Comp} \rightarrow \text{Comp}$ admits a tensor product, then the equality $\text{AR}[F] = \text{AE}(0)$ holds if and only if $\{0, 1\} \in \text{AR}[F]$.*

Since each monadic normal functor admit a tensor product (see [23, §3.4]), this corollary implies:

Corollary 3.11. *For a monadic normal functor $F : \text{Comp} \rightarrow \text{Comp}$ the equality $\text{AR}[F] = \text{AE}(0)$ holds if and only if $\{0, 1\} \in \text{AR}[F]$.*

Applying Corollary 3.11 to the functor of idempotent measures I (which monadic and normal [26]), we get the following corollary answering a problem posed in [1].

Corollary 3.12. *The functor of idempotent measures I has $\text{AR}[I] = \text{AE}(0)$.*

For non-normal functors F the class $\text{AR}[F]$ need not be contained in the class $\text{AE}(0)$ of Dugundji compacta. A typical example is the functor of superextension λ , see [11].

Theorem 3.13 (Ivanov). *The class $\text{AR}[\lambda]$ of absolute λ -valued retracts coincides with the class of openly generated connected compacta.*

This fact was crucial in the proof of the coincidence of the class OG of openly generated compacta with the class κM of κ -metrizable compacta, see [22]. According to [21] and [22], κ -metrizable compacta can be characterized as follows:

Theorem 3.14 (Shirokov, Ščepin). *A compact Hausdorff space (X, τ_X) is κ -metrizable if and only if it is openly generated if and only if for any embedding $X \subset Y$ into a compact Hausdorff space (Y, τ_Y) there is an operator $e : \tau_X \rightarrow \tau_Y$ such that*

- $e(U) \cap X = U$ for all open sets $U \subset X$ and
- $e(U) \cap e(V) = \emptyset$ for any disjoint open sets $U, V \subset X$.

4. RECOGNIZING ABSOLUTE F-VALUED RETRACTS FOR SOME FUNCTIONAL FUNCTORS

In this section we shall study the class $\text{AR}[F]$ for functional functors F . First we recall some properties of functionals.

Let X be a compact space. We shall say that a functional $\mu : C(X) \rightarrow \mathbb{R}$

- *preserves constants* if $\mu(c_X) = c$ for any constant function $c_X : X \rightarrow \{c\} \subset \mathbb{R}$;
- *preserves order* if $\mu(f) \leq \mu(g)$ for any functions $f \leq g$ in $C(X)$;
- *weakly preserves order* if $\mu(a) \leq \mu(f) \leq \mu(b)$ for any function $f \in C(X)$ and constant functions $a, b \in C(X)$ with $a \leq f \leq b$;

- *preserves minima* if $\mu(\min\{f, g\}) = \min\{\mu(f), \mu(g)\}$ for any functions $f, g \in C(X)$;
- *preserves maxima* if $\mu(\max\{f, g\}) = \max\{\mu(f), \mu(g)\}$ for any functions $f, g \in C(X)$;
- *weakly preserves minima* if $\mu(\min\{f, c\}) = \min\{\mu(f), \mu(c)\}$ for any $f \in C(X)$ and any constant function $c \in C(X)$;
- *weakly preserves maxima* if $\mu(\max\{f, c\}) = \max\{\mu(f), \mu(c)\}$ for any $f \in C(X)$ and any constant function $c \in C(X)$;
- is *additive* if $\mu(f + g) = \mu(f) + \mu(g)$ for any functions $f, g \in C(X)$;
- is *weakly additive* if $\mu(f + c) = \mu(f) + \mu(c)$ for any $f \in C(X)$ and any constant function $c \in C(X)$;
- is *weakly multiplicative* if $\mu(c \cdot f) = \mu(c) \cdot \mu(f)$ for any $f \in C(X)$ and any constant function $c \in C(X)$;
- is *k-Lipschitz* for $k \geq 1$ if $|\mu(f) - \mu(g)| \leq k \cdot \|f - g\|$ for any functions $f, g \in C(X)$.

Here for a function $f \in C(X)$ by $\|f\| = \sup_{x \in X} |f(x)|$ we denote its norm in the Banach space $C(X)$.

These properties of functionals allow us to define subfunctors of the functor $\mathbb{R}^{C(\cdot)}$ assigning to each compact space X the following closed subspaces of the functional space $\mathbb{R}^{C(X)}$:

- $\overline{V}(X) = \{\mu \in \mathbb{R}^{C(X)} : \mu \text{ preserves constants}\}$;
- $V_{<}(X) = \{\mu \in \overline{V}(X) : \mu \text{ preserves order}\}$;
- $V_{\square}(X) = \{\mu \in \overline{V}(X) : \mu \text{ weakly preserves order}\}$;
- $V_{\min}(X) = \{\mu \in \overline{V}(X) : \mu \text{ preserves minima}\}$;
- $V_{\max}(X) = \{\mu \in \overline{V}(X) : \mu \text{ preserves maxima}\}$;
- $V_{\underline{\min}}(X) = \{\mu \in \overline{V}(X) : \mu \text{ weakly preserves minima}\}$;
- $V_{\underline{\max}}(X) = \{\mu \in \overline{V}(X) : \mu \text{ weakly preserves maxima}\}$;
- $V_{+}(X) = \{\mu \in \overline{V}(X) : \mu \text{ is additive}\}$;
- $V_{\pm}(X) = \{\mu \in \overline{V}(X) : \mu \text{ is weakly additive}\}$;
- $V_{\cdot}(X) = \{\mu \in \overline{V}(X) : \mu \text{ is weakly multiplicative}\}$;
- $\text{Lip}_k(X) = \{\mu \in \overline{V}(X) : \mu \text{ is } k\text{-Lipschitz}\}$ for $k \geq 1$.

Many known functors can be written as intersections of the above functors. For example,

- $V = V_{\square}$ is the universal functor considered by T.Radul [?];
- $P = V_{+} \cap V_{\cdot} \cap V_{<}$ is the functor of probability measures (see [8]);
- $I = V_{\pm} \cap V_{\max}$ is the functor of idempotent measures (see [26]);
- $O = V_{\pm} \cap V_{<}$ is the functor of weakly additive order-preserving functionals introduced by T.Radul [?];
- $S = V_{\cdot} \cap V_{\pm} \cap V_{<}$ is the functor introduced and studied by V.Valov [25];
- $V_{\underline{\max}} \cap V_{\underline{\min}} \cap V_{<}$ is isomorphic to the functor G of growth hyperspaces (see [?]);
- $V_{\underline{\min}} \cap V_{\max} \cap V_{\pm}$ and $V_{\min} \cap V_{\underline{\max}} \cap V_{\pm}$ are isomorphic to the hyperspace functor \exp (see [?]).

Since the functors $V_{\underline{\min}} \cap V_{\max} \cap V_{\pm}$ and $V_{\min} \cap V_{\underline{\max}} \cap V_{\pm}$ are isomorphic to the hyperspace functor \exp , Fedorchuk's description [7] of the class $\text{AR}[\exp]$ implies:

Theorem 4.1. $\text{AR}[V_{\underline{\min}} \cap V_{\max} \cap V_{\pm}] = \text{AR}[V_{\min} \cap V_{\underline{\max}} \cap V_{\pm}] = \text{AR}[\exp] = \text{AE}(1)$.

The following description of the class $\text{AR}[S]$ for the functor $S = \overline{V} \cap V_{\cdot} \cap V_{\pm} \cap V_{<}$ was obtained by V.Valov in [25].

Theorem 4.2 (Valov). *For the functor $S = V_{\cdot} \cap V_{\pm} \cap V_{<}$ the class $\text{AR}[S]$ coincides with the class OG of openly generated compacta.*

In fact, the inclusion $\text{AR}[S] \subset \text{OG}$ can be derived from the following general theorem.

Theorem 4.3. *A compact space X is openly generated if $X \in \text{AR}[F]$ for some functional functor F such that*

$$FX \subset \{\mu \in \overline{V}(X) : \sup\{|\mu(f) - \mu(g)| : f, g \in C(X), f \cdot g = 0, \|f - g\| \leq 1\} < 2\}.$$

Proof. Fix any compact space $X \in \text{AR}[F]$. If X is empty, then there is nothing to prove. So, we assume that X is not empty. To show that X is openly generated, we shall apply the characterization Theorem 3.14. Embed X into any compact Hausdorff space Y . Since X is an absolute F -valued retract, there is a map $r : Y \rightarrow FX$ such that $r(x) \in F(\{x\}) \subset FX$ for every $x \in X$. This implies that $F(\{x\}) \subset \overline{V}(x)$ is not empty and hence coincides with the singleton $\overline{V}(\{x\}) = \{\delta_X(x)\}$. This allows us to identify X with the closed subset $\{\delta_X(x) : x \in X\} \subset FX \subset \overline{V}(X)$ consisting of Dirac measures.

First we construct an extension operator $e : \tau_X \rightarrow \tau_{FX}$. Choose a positive $\varepsilon > 0$ such that

$$2 - 2\varepsilon > \sup\{|\mu(f) - \mu(g)| : f, g \in C(X), fg = 0, \|f - g\| \leq 1\}.$$

Given an open set $U \subset X$ consider the open subset $e(U)$ of FX that consists of all functionals $\mu \in FX \subset \mathbb{R}^{C(X)}$ for which there is a continuous function $f : X \rightarrow [0, 1]$ such that $\mu(f) > 1 - \varepsilon$, $\mu(-f) < -1 + \varepsilon$ and $\text{supp}(f) = f^{-1}(0, 1] \subset U$. The complete regularity of X implies that $e(U) \cap X = U$.

We claim that $e(U) \cap e(V) = \emptyset$ for any disjoint open sets $U, V \subset X$. Assuming the converse, we can find a functional $\mu \in e(U) \cap e(V)$ and two continuous functions $f_U, f_V : X \rightarrow [0, 1]$ such that

- (1) $\text{supp}(f_U) \subset U$, $\text{supp}(f_V) \subset V$;
- (2) $\mu(f_U) > 1 - \varepsilon$ and $\mu(-f_V) < -1 + \varepsilon$.

The condition (1) implies that $f_U \cdot (-f_V) = 0$, $\|f_U + f_V\| \leq 1$ and hence

$$|\mu(f_U) - \mu(-f_V)| \leq 2 - 2\varepsilon$$

by the choice of ε . On the other hand, the condition (2) yields that $|\mu(f_U) - \mu(-f_V)| > 2 - 2\varepsilon$ and this is a desired contradiction.

The operator $e : \tau_X \rightarrow \tau_{FX}$ and the map $r : Y \rightarrow FX$ generate the operator

$$r^{-1} \circ e : \tau_X \rightarrow \tau_Y, \quad r^{-1} \circ e : U \mapsto r^{-1}(e(U)),$$

which has two properties from Theorem 3.14. This theorem implies that the space X is openly generated. \square

Combining Theorem 4.3 with Valov's Theorem 4.2, we get:

Corollary 4.4. *For any $k \in [1, 2)$ and any subfunctor $F \subset \text{Lip}_k$ we get $\text{AR}[F] \subset \text{OG}$. If, moreover, $V_{\leq} \cap V_{\pm} \cap V_{<} \subset F$, then $\text{AR}[F] = \text{OG}$.*

This corollary implies the following result first established by Valov [25].

Corollary 4.5. *The functor $O = V_{\pm} \cap V_{<}$ of weakly additive order-preserving functionals has $\text{AR}[O] = \text{OG}$.*

It is interesting to note that for $k \geq 3$ Corollary 4.4 does not hold because the class $\text{AR}[\text{Lip}_3]$ contains some scattered compact spaces, which are not openly generated.

Let us recall that a topological space X is called *scattered* if each non-empty subspace A of X contains an isolated point. For a subspace A of X by $A^{(1)}$ we denote the set of non-isolated points of A and for every ordinal α define the α -th derived set $X^{(\alpha)}$ of X letting $X^{(0)} = X$, $X^{(1)}$ be the set of non-isolated points of X and

$$X^{(\alpha)} = \bigcap_{\beta < \alpha} (X^{(\beta)})^{(1)}$$

for $\alpha > 1$. For a scattered space X the transfinite sequence $(X_{\alpha})_{\alpha}$ is strictly decreasing, so $X^{(\alpha)} = \emptyset$ for some α . The *scattered height* $\text{ht}(X)$ of X is the smallest ordinal α such that the α th derived set $X^{(\alpha)}$ is finite. For example, an infinite compact space X with a unique non-isolated point has scattered height $\text{ht}(X) = 1$.

A topological space X is called *hereditarily paracompact* if each subspace of X is paracompact. By [3], a topological space is compact scattered and hereditarily paracompact if and only if it belongs to the smallest class of spaces containing the singletons and closed under taking one-point compactifications of topological sums of spaces from the class.

Theorem 4.6. *Each hereditarily paracompact scattered compact space X of finite scattered height $s = \text{ht}(X)$ with $|X^{(s)}| = 1$ is an absolute F_s -valued retract for the functor $F_s = V_{\leq} \cap V_{\pm} \cap V_{<} \cap \text{Lip}_{2^{s+1}-1}$.*

Proof. Since $V_{\leq} X = VX = \prod_{f \in C(X)} [\min f, \max f]$ is an absolute retract, it suffices to construct a continuous map $p_X : VX \rightarrow F_s(X)$ such that $p_X \circ \delta_X = \delta_X$. The existence of such map p_X will be proved by induction on the scattered height of $s = \text{ht}(X)$. If $s = 0$, then $X^{(0)} = X$ is a singleton and so is the space $F_0 X$. Then the constant map $r_X : VX \rightarrow F_0 X$ has the desired property: $p_X \circ \delta_X = \delta_X$.

Now we assume that for some $s \geq 1$ the existence of a map $p_K : VK \rightarrow F_{\text{ht}(K)} K$ with $p_K \circ \delta_K = \delta_K$ has been proved for all hereditarily paracompact scattered compacta K such that $\text{ht}(K) < s$ and $|K^{(\text{ht}(K))}| = 1$. Assume that X is a hereditarily paracompact scattered compact space of scattered height $\text{ht}(X) = s$ and $X^{(s)}$ is a singleton $\{*\}$. By a result of Telgarsky [24], the scattered space $X \setminus \{*\}$, being paracompact, is strongly zero-dimensional, which allows us to find a disjoint cover \mathcal{A} of $X \setminus \{*\}$ by non-empty compact open subsets of $X \setminus \{*\}$. Each space $A \in \mathcal{A}$ has scattered height $\text{ht}(A) < \text{ht}(X) = s$. Decomposing A into a disjoint finite union of open subsets, we can additionally assume that $A^{(\text{ht}(A))}$ is a singleton for every $A \in \mathcal{A}$. The inductive assumption yields a continuous map $p_A : VA \rightarrow F_{\text{ht}(A)}(A) \subset F_{s-1}(A)$ such that $p_A \circ \delta_A = \delta_A$. Let $\chi_A : X \rightarrow \{0, 1\}$ denote the characteristic function of the set A (which means that $A = \chi_A^{-1}(1)$). Fix any retraction $r_A : X \rightarrow A$ and consider the retraction $Vr_A : VX \rightarrow VA$. For a functional $\mu \in VX$ put $\mu_A = p_A \circ Vr_A(\mu) \in F_{s-1}(A)$. Let $\mathcal{A}^0 = \mathcal{A} \cup \{\emptyset\}$.

Define the map $p_X : VX \rightarrow \overline{V}X$ assigning to each functional $\mu \in VX$ the functional $p_X(\mu) \in \overline{V}X$, which assigns to each function $\varphi \in C(X)$ the real number

$$p_X(\mu)(\varphi) = \varphi(*) + \sup_{A \in \mathcal{A}^\circ} \mu(\chi_A) \cdot \mu_A(\varphi|A - \varphi(*)) + \inf_{A \in \mathcal{A}^\circ} \mu(\chi_A) \cdot \mu_A(\varphi|A - \varphi(*)).$$

In this formula for $A = \emptyset$ we assume that $\mu(\chi_A) \cdot \mu_A(\varphi|A - \varphi(*)) = 0$, which implies that

$$\sup_{A \in \mathcal{A}^\circ} \mu(\chi_A) \cdot \mu_A(\varphi|A - \varphi(*)) = \max_{A \in \mathcal{A}^\circ} \mu(\chi_A) \cdot \mu_A(\varphi|A - \varphi(*)) \geq 0$$

and

$$\inf_{A \in \mathcal{A}^\circ} \mu(\chi_A) \cdot \mu_A(\varphi|A - \varphi(*)) = \min_{A \in \mathcal{A}^\circ} \mu(\chi_A) \cdot \mu_A(\varphi|A - \varphi(*)) \leq 0.$$

The definition of the functional $p_X(\mu)$ implies that it is weakly additive. The weak multiplicativity of the functionals μ_A , $A \in \mathcal{A}$, implies the weak multiplicativity of the functional $p_X(\mu)$. Next, we prove that the functional $p_X(\mu)$ weakly preserves order. Indeed,

$$\begin{aligned} p_X(\mu)(\varphi) &= \varphi(*) + \sup_{A \in \mathcal{U}} \mu(\chi_A) \cdot \mu_A(\varphi|A - \varphi(*)) + \inf_{A \in \mathcal{U}} \mu(\chi_A) \cdot \mu_A(\varphi|A - \varphi(*)) \geq \\ &\geq \varphi(*) + 0 + \inf_{A \in \mathcal{U}} \mu(\chi_A) (\min \varphi|A - \varphi(*)) \geq \min \varphi|A \geq \min \varphi. \end{aligned}$$

By analogy we can prove that $p_X(\mu)(\varphi) \leq \max \varphi$. So, $p_X(\mu) \in (V_- \cap V_\pm \cap V_+)(X)$.

Finally, let us check that $p_X(\mu) \in \text{Lip}_{2^{s+1}-1}(X)$. Fix any two functions $\varphi, \psi \in C(X)$ and find (possibly empty) sets $A^\varphi, A_\varphi, A^\psi, A_\psi \in \mathcal{U}$ such that

$$\begin{aligned} \mu(\chi_{A^\varphi}) \cdot \mu_{A^\varphi}(\varphi|A^\varphi - \varphi(*)) &= \sup_{A \in \mathcal{U}} \mu(\chi_A) \cdot \mu_A(\varphi|A - \varphi(*)), \\ \mu(\chi_{A_\varphi}) \cdot \mu_{A_\varphi}(\varphi|A_\varphi - \varphi(*)) &= \inf_{A \in \mathcal{U}} \mu(\chi_A) \cdot \mu_A(\varphi|A - \varphi(*)), \\ \mu(\chi_{A^\psi}) \cdot \mu_{A^\psi}(\psi|A^\psi - \psi(*)) &= \sup_{A \in \mathcal{U}} \mu(\chi_A) \cdot \mu_A(\psi|A - \psi(*)), \\ \mu(\chi_{A_\psi}) \cdot \mu_{A_\psi}(\psi|A_\psi - \psi(*)) &= \inf_{A \in \mathcal{U}} \mu(\chi_A) \cdot \mu_A(\psi|A - \psi(*)). \end{aligned}$$

The weak additivity and the $(2^s - 1)$ -Lipschitz property of the functionals μ_A , $A \in \mathcal{A}$, imply:

$$\begin{aligned} p_X(\mu)(\varphi) - p_X(\mu)(\psi) &= \varphi(*) + \mu(\chi_{A^\varphi}) \cdot \mu_{A^\varphi}(\varphi|A^\varphi - \varphi(*)) + \mu(\chi_{A_\varphi}) \cdot \mu_{A_\varphi}(\varphi|A_\varphi - \varphi(*)) - \\ &\quad - \psi(*) - \mu(\chi_{A^\psi}) \cdot \mu_{A^\psi}(\psi|A^\psi - \psi(*)) - \mu(\chi_{A_\psi}) \cdot \mu_{A_\psi}(\psi|A_\psi - \psi(*)) \leq \\ &\leq \varphi(*) + \mu(\chi_{A^\varphi}) \cdot \mu_{A^\varphi}(\varphi|A^\varphi - \varphi(*)) + \mu(\chi_{A_\varphi}) \cdot \mu_{A_\varphi}(\varphi|A_\varphi - \varphi(*)) - \\ &\quad - \psi(*) - \mu(\chi_{A^\psi}) \cdot \mu_{A^\psi}(\psi|A^\psi - \psi(*)) - \mu(\chi_{A_\psi}) \cdot \mu_{A_\psi}(\psi|A_\psi - \psi(*)) = \\ &= (1 - \mu(\chi_{A^\varphi}) - \mu(\chi_{A_\psi}))(\varphi(*) - \psi(*)) + \\ &\quad + \mu(\chi_{A^\varphi}) \cdot (\mu_{A^\varphi}(\varphi|A^\varphi) - \mu_{A^\varphi}(\psi|A^\varphi)) + \\ &\quad + \mu(\chi_{A_\varphi}) \cdot (\mu_{A_\varphi}(\varphi|A_\varphi) - \mu_{A_\varphi}(\psi|A_\varphi)) \leq \\ &\leq (1 - \mu(\chi_{A^\varphi}) - \mu(\chi_{A_\psi}))\|\varphi - \psi\| + (2^s - 1)\|\varphi - \psi\| + (2^s - 1)\|\varphi - \psi\| \leq \\ &\leq (1 + 2(2^s - 1))\|\varphi - \psi\| = (2^{s+1} - 1)\|\varphi - \psi\|. \end{aligned}$$

By analogy we can prove that $p_X(\mu)(\psi) - p_X(\mu)(\varphi) \leq (2^{s+1} - 1)\|\varphi - \psi\|$, which implies that the functional $p_X(\mu)$ is $(2^{s+1} - 1)$ -Lipschitz and hence belongs to $\text{Lip}_{2^{s+1}-1}(X)$.

Using the continuity of the maps p_A , $A \in \mathcal{A}$, and the fact that for any $\varepsilon > 0$ the norm $\|\varphi|A - \varphi(*)\| < \varepsilon$ for all but finitely many sets $A \in \mathcal{A}$, we can show that the map $p_X : VX \rightarrow \overline{V}(X)$ is continuous.

Finally, we check that $p_X(\mu) = \mu$ if $\mu = \delta_X(x)$ is the Dirac measure at a point $x \in X$. If $x = *$, then $\mu(\chi_A) = \chi_A(*) = 0$ and hence $p_X(\mu)(\varphi) = \varphi(*) = \mu(\varphi)$ for any function $\varphi \in C(X)$. So, we assume that $x \in X \setminus \{*\}$. Since \mathcal{A} is a disjoint cover of X , there is a unique set $A \in \mathcal{A}$ containing x . It follows that $r_A(x) = x$ and hence $\mu_A = p_A \circ V r_A(\mu) = p_A \circ V r_A(\delta_X(x)) = p_A(\delta_A(x)) = \delta_A(x)$. Take any function $\varphi \in C(X)$. If $\varphi(x) \geq \varphi(*)$, then

$$p_X(\mu)(\varphi) = \varphi(*) + \mu(\chi_A) \cdot \mu_A(\varphi|A - \varphi(*)) + 0 = \varphi(*) + 1 \cdot \delta_A(x)(\varphi|A - \varphi(*)) = \varphi(*) + (\varphi(x) - \varphi(*)) = \varphi(x).$$

If $\varphi(x) \leq \varphi(*)$, then

$$p_X(\mu)(\varphi) = \varphi(*) + 0 + \mu(\chi_A) \cdot \mu_A(\varphi|A - \varphi(*)) = \varphi(*) + 1 \cdot (\varphi(x) - \varphi(*)) = \varphi(x).$$

So, $p_X(\mu) = p_X(\delta_X(x)) = \delta_X(x) = \mu$. □

Theorem 4.7. *Each hereditarily paracompact scattered compact space X of finite scattered height $s = \text{ht}(X)$ is an absolute F -valued retract for the functor $F_{s+1} = V_- \cap V_\pm \cap V_+ \cap \text{Lip}_{2^{s+2}-1}$.*

Proof. Consider the one-point compactification Y of the product $X \times \mathbb{N}$ of X with the countable discrete space \mathbb{N} and observe that $\text{ht}(Y) = \text{ht}(X) + 1 = s + 1$ and $Y^{(s+1)}$ is a singleton. By Theorem 4.6, the space Y is an absolute F -valued retract for the functor $F_{s+1} = V_{\perp} \cap V_{\pm} \cap V_{\square} \cap \text{Lip}_{2^{s+2}-1}$ and so is the space X , being homeomorphic to a retract of Y . \square

It is known [22, 1.7] that openly generated compacta have countable cellularity. This implies that any uncountable compact space X with a unique non-isolated point is not openly generated. On the other hand, it is an absolute F -valued retract for the functor $F = V_{\perp} \cap V_{\pm} \cap V_{\square} \cap \text{Lip}_3$ according to Theorem 4.6. Let us write this fact for future references.

Corollary 4.8. *Any uncountable compact space X with a unique non-isolated point is not openly generated but is an absolute F -valued retract for the functor $F = V_{\perp} \cap V_{\pm} \cap V_{\square} \cap \text{Lip}_3$. For this functor we get $\text{AR}[F] \not\subseteq \text{OG}$.*

Theorems 4.3, 4.6 and Proposition 3.5 have an interesting corollary.

Corollary 4.9. *The functor $V_{\perp} \cap V_{\pm} \cap V_{\square} \cap \text{Lip}_3$ admits no natural transformation into the functor Lip_k for any $k \in [1, 2)$.*

By the same reason, Theorems 3.6 and 4.2 imply:

Corollary 4.10. *The functor $S = V_{\perp} \cap V_{\pm} \cap V_{\triangleleft}$ admits no natural transformation into a normal functor $F : \text{Comp} \rightarrow \text{Comp}$.*

It is interesting to note that the functor $F = V_{\perp} \cap V_{\pm} \cap V_{\square}$ has maximal possible class $\text{AR}[F]$ of absolute F -valued retracts.

Theorem 4.11. *For the functor $F = V_{\perp} \cap V_{\pm} \cap V_{\square}$ the class $\text{AR}[F]$ coincides with the class of all compact Hausdorff spaces.*

Proof. Observe that for every compact space X the space $V_{\square}X = VX = \prod_{\varphi \in C(X)} [\min \varphi, \max \varphi]$ is an absolute retract, which implies that the class $\text{AR}[V]$ contains all compact spaces. To prove that $\text{AR}[F] = \text{AR}[V]$ it suffices to construct a retraction $r_X : VX \rightarrow FX$.

Consider the functional $\alpha \in FX$ assigning to each function $\varphi \in C(X)$ the real number $\alpha(\varphi) = \frac{1}{2}(\min \varphi + \max \varphi)$. Define the retraction $r_X : VX \rightarrow FX$ assigning to each functional $\mu \in V_{\square}(X)$ the functional $p_X(\mu)$ assigning to each non-constant function $\varphi \in C(X)$ the real number

$$\alpha(\varphi) + \frac{1}{2} \|\varphi - \alpha(\varphi)\| \cdot \left(\mu \left(\frac{\varphi - \alpha(\varphi)}{\|\varphi - \alpha(\varphi)\|} \right) - \mu \left(-\frac{\varphi - \alpha(\varphi)}{\|\varphi - \alpha(\varphi)\|} \right) \right).$$

It can be shown that $r_X(\mu) \in FX$ and the map $r_X : VX \rightarrow FX$ is a well-defined retraction of VX onto FX . \square

5. SOME OPEN PROBLEMS

Problem 5.1. *Is the functor $F = V_{\perp} \cap V_{\pm} \cap V_{\square}$ a (natural) retract of the functor V_{\square} ?*

It seems that the retractions $r_X : VX \rightarrow FX$ constructed in the proof of Theorem 4.11 do not determine a natural transformation $r : V \rightarrow F$.

5.1. Absolute Lip_k -valued retracts.

Problem 5.2. *Is any compact space X with a unique non-isolated point an absolute Lip_2 -valued retract?*

Problem 5.3. *Is each (scattered) compact space an absolute Lip_k -valued retract for some k ?*

The answer to this problem is unknown even for the Mrówka space $\psi_{\mathcal{A}}(\mathbb{N})$ generated by an uncountable almost disjoint family \mathcal{A} of infinite subsets of \mathbb{N} . By definition, $\psi_{\mathcal{A}}(\mathbb{N})$ is the Stone space of the Boolean algebra generated by $\mathcal{A} \cup \{\{n\} : n \in \mathbb{N}\}$. In other words, $\psi_{\mathcal{A}}(\mathbb{N})$ is the one-point compactification of the locally compact space $\mathbb{N} \cup \mathcal{A}$ in which all points $n \in \mathbb{N}$ are isolated and for any $A \in \mathcal{A}$ the family

$$\mathcal{B}_A = \{\{A\} \cup A \setminus F : F \text{ is a finite subset of } \mathbb{N}\}$$

is a neighborhood base at A . The Mrówka space has scattered height $\text{ht}(\psi_{\mathcal{A}}(\mathbb{N})) = 2$ but is not hereditarily paracompact. It is separable but contains an uncountable discrete subspace. If the almost disjoint family \mathcal{A} is maximal, then $\psi_{\mathcal{A}}(\mathbb{N})$ is sequential but not Fréchet-Urysohn.

Problem 5.4. *Is the Mrówka space $\psi_{\mathcal{A}}(\mathbb{N})$ an absolute Lip_k -valued retract for some real $k \geq 1$?*

We say that a functor $F : \text{Comp} \rightarrow \text{Comp}$ is *weight-preserving* if for any infinite compact space X the weight of the space FX coincides with the weight of X .

Problem 5.5. *Can each compact space be absolute F -valued retract for some weight-preserving subfunctor $F \subset \overline{V}$? Is it true for the functor Lip_3 ?*

Problem 5.6. *Is the Stone-Ćech compactification $\beta\mathbb{N}$ of positive integers an absolute Lip_3 -valued retract? Is the remainder $\beta\mathbb{N} \setminus \mathbb{N}$ of $\beta\mathbb{N}$ an absolute Lip_3 -valued retract?*

5.2. Absolute F -valued retracts for functors with finite supports. A functor $F : \text{Comp} \rightarrow \text{Comp}$ is defined to have *finite supports* if for each compact space X and an element $a \in FX$ there is a map $f : A \rightarrow X$ for a finite space A such that $a \in Ff(FA) \subset FX$.

Problem 5.7. *Can a functor $F : \text{Comp} \rightarrow \text{Comp}$ with finite supports have $\text{AR}[F] \supset \text{AE}(0)$?*

For a functor $F : \text{Comp} \rightarrow \text{Comp}$ and a natural number $n = \{0, \dots, n-1\}$ let F_n be a subfunctor of F assigning to each compact space X the subspace

$$F_n X = \{a \in FX : \exists f \in C(n, X) \text{ such that } a \in Ff(Fn)\}.$$

Problem 5.8. *Given a functor $F : \text{Comp} \rightarrow \text{Comp}$ and a natural number n study the class $\text{AR}[F_n]$.*

The class of absolute P_n -valued retracts has been studied in [4] where it was observed that each n -dimensional compact metrizable space is an absolute P_{n+2} -valued retract and was proved that each n -dimensional hereditarily indecomposable compact space is not an absolute P_n -valued retract.

5.3. Openly generated compacta and the superextension functor. The following problem was motivated by Ivanov's Theorem 3.13.

Problem 5.9. *Let $F : \text{Comp} \rightarrow \text{Comp}$ be a monadic functional functor such that each connected openly generated compact Hausdorff space is an absolute F -valued retract. Is there a natural transformation $\lambda \rightarrow F$ from the functor of superextension λ into F ?*

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